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Splicing of Markov and Weak Markov Systems

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INTRODUCTION

We consider here the splicing of two normalized Markov systems or normalized weak Markov systems defined on overlapping sets. After introducing necessary terminology, we state and prove our results. Last of all, we show by way of an example what can happen if certain of our hypotheses are relaxed.

TERMINOLOGY

We will refer to any sequence of real-valued functions as a *system.* Given a set A of real numbers of cardinality at least $n+2$ (n finite $\geqslant 0$), a system $Y_n = \{y_i\}_{i=0}^n$ of real-valued functions defined on A will be called a *Tchebycheff system* on A, provided that, for every sequence $\{t_0, ..., t_n\}$ of points from A such that $t_0 < \cdots < t_n$, the determinant $\det(y_i(t_j))_{i=0}^n$ is strictly positive. If the same determinant is merely nonnegative and Y_n is linearly independent on A, Y_n will be referred to as a *weak Tchebycheff system on A*. The system Y_n will be called a *Markov system* (respectively, a weak Markov system) on A, provided that $Y_k := \{y_i\}_{i=0}^k$ is a Tchebycheff system (respectively, a weak Tchebycheff system) on A for each $k = 0, ..., n$. The system Y_n is said to be *normalized,* if y_0 is the constant function 1 in which case we set (if $n \geqslant 1$) $Y'_n = \{y'_1, ..., y'_n\}$.

A real set A is said to have property (B) if, for every two (distinct) points in A, there is a point in A between them.

Let Y_n and Z_n be two systems. We will say that Y_n is obtainable from Z_n by a *triangular transformation* if $y_0 = z_0$ and $y_k - z_k$ lie in the linear span of Z_{k-1} for $k = 1, ..., n$.

A weak Markov system Z_n defined on a set A will be called *weakly nondegenerate* if it satisfies Condition I and Condition E as stated below.

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CONDITION I. For every real number c, Z_n is linearly independent on at least one of the sets $(c, \infty) \cap A$ and $(-\infty, c) \cap A$.

CONDITION E. For every point c in the convex hull of A:

(i) If Z_n is linearly independent on $[c, \infty) \cap A$, then there exists a set U_n, obtainable from Z_n by a triangular linear transformation, such that for any subsequence $\{k(r): r = 0, ..., m\}$ of the sequence $\{0, ..., n\}$, the set $\{u_{k(0)}, ..., u_{k(m)}\}$ is a weak Markov system on $[c, \infty) \cap A$; and

(ii) If Z_n is linearly independent on $(-\infty, c] \cap A$, then there exists a set V_n, obtainable from Z_n by a triangular transformation, such that for any subsequence $\{k(r): r = 0, ..., m\}$ of the sequence $\{0, ..., n\}$, the set $\{(-1)^{0-k(0)} v_{k(0)}, ..., (-1)^{m-k(m)} v_{k(m)}\}$ is a weak Markov system on $(-\infty, c] \cap A$.

Remark 1. If A consists of at least $2n+3$ points and Z_n is assumed from the outset to be a Markov system, Condition I is redundant, and Condition E may be stated more simply. Condition E is also implied if Z_n is a Markov system and its underlying set A has property (B) and contains neither its supremum nor its infimum (this assertion readily follows from, e.g., [10, Theorem 1 and Corollary 2]).

Remark 2. The above definition of weak nondegeneracy is introduced in Zalik [9], where it is shown among other things that if a normalized weak Markov system is weakly nondegenerate then the functions in it can be represented by means of iterated Riemann–Stieltjes integrals.

A system of functions Z_n defined on a set A will be called *C-bounded* if each function in Z_n is bounded on the intersection of A with any compact subset of the convex hull of A (which we shall denote by $I(A)$); if the set A is an interval and every element of Z_n is absolutely continuous in any closed subinterval of A, we will say that Z_n is *C-absolutely continuous.*

Remark 3. It is a consequence of [9, Lemma 3] that every weakly nondegenerate normalized weak Markov system is C-bounded.

RESULTS

THEOREM 1. *Let U_n and V_n be weakly nondegenerate normalized weak Markov systems defined respectively on sets A and B, with U_n (or V_n) linearly independent on $A \cap B$, and with $u_k(t) = v_k(t)$ for every t in $A \cap B$ and $k = 0, ..., n$. Assume further that for every point c in $A \cap B$, every point of A to the right of c lies in B, and every point in B to the left of c lies in A. Then, if a set of functions Z_n is defined by $z_k(t) = u_k(t)$ for $t \in A$ and $z_k(t) = v_k(t)$*

for $t \in B$, $k = 0, ..., n$, *the set* Z_n *is a weakly nondegenerate weak Markov system on* $A \cup B$.

As a consequence of Theorem 1, we have:

THEOREM 2. *Assume that the sets* A *and* B *both satisfy property* (B) *and that neither of them has a first nor a least element. Let* U_n *and* V_n *be normalized Markov systems defined respectively on* A *and* B, *with* $A \cap B \neq \varnothing$ *and* $u_k(t) = v_k(t)$ *for every* t *in* $A \cap B$ *and* $k = 0, ..., n$. *Assume further that for every point* c *in* $A \cap B$, *every point of* A *to the right of* c *lies in* B, *and every point in* B *to the left of* c *lies in* A. *Then, if a set of functions* Z_n *is defined by* $z_k(t) = u_k(t)$ *for* $t \in A$ *and* $z_k(t) = v_k(t)$ *for* $t \in B$, *the set* Z_n *is a Markov system on* $A \cup B$.

A version of Theorem 2 for continuous Markov systems defined on intervals was apparently first stated by Rutman [3] (see also Krein and Nudel'man [2, p. 50, Cor. 2]). However, this result was based on an integral representation for Markov systems (also laid down by Rutman in [3]) which was shown, independently by Zalik [6] and Zielke (cf. [12]), to be erroneous. The hypotheses of Theorem 2 imply that the systems U_n and V_n overlap at an infinite number of points. One could say alternatively that, as in Theorem 1, the systems U_n and V_n are both linearly independent on the intersection of the underlying sets A and B. If, however, the overlap occurs at only one point, one may see from the example at the end of this communication that the conclusion of Theorem 2 is no longer valid. A result somewhat related to Theorem 2 is due to Bartelt [1] (see also Schumaker [4, p. 473, Theorem 11.12]).

To prove Theorem 1, we need the following result which generalizes Lemma 6 of [9]. The proof given there suffices here also with only a minor change and therefore will not be repeated.

THEOREM 3. *Let* U_n *be a weakly nondegenerate weak Markov system defined on a set* A. *If, for some point* c *in* A, $u_0(c) = 0$, *then* $u_k(c) = 0$ *for* $k = 0, ..., n$.

To prove Theorem 2, we shall use Theorem 1 and the following generalization of the theorem of [7], the proof of which will be omitted here because it would differ in no essential respect from that given in [7]. A system of functions U_n defined on a set A with property (B) will be called "substantial" provided that for every a and b in A with $a < b$, U_n is linearly independent on $[a, b] \cap A$.

We have:

THEOREM 4. *Let* U_n *be a system of functions defined on a set* A *having* *property* (B) *and containing neither its supremum nor its infimum. Then the following propositions are equivalent*:

(a) *The system* U_n *is a substantial weak Tchebycheff system on* A *and* $u_0(t) > 0$ *on* A.

(b) *The system* U_n *is a substantial weak Tchebycheff system on* A, *and, for every point* c *in* A *not all the functions in* U_n *vanish in* A.

(c) *The system* U_n *is a Tchebycheff system on* A.

Proofs

Before giving the proof of Theorem 1, we will state two preliminary results. Particularly useful as a starting point in the proof of Theorem 1 is Lemma 1, which is a refinement of [9, Theorem 3], which in turn is a refinement of the theorem of [8]. Before stating Lemma 1, however, we need to give the following definition:

Let $X_n = \{x_0, ..., x_n\}$ be a set of real-valued functions defined on a set $A \subseteq R$ and let $Y_n = \{y_0, ..., y_n\}$ be a set of real-valued functions defined on $B \subseteq R$. We say that X_n can be *embedded* in Y_n if there is a strictly increasing function $h: A \to B$ such that $y_i[h(t)] = x_i(t)$ for every $t \in A$ and $i = 0, 1, ..., n$. The function h is called an *embedding function.*

Lemma 1. *Let* U_n *and* V_n *be weakly nondegenerate normalized weak Markov systems defined respectively on sets* A *and* B *such that* $A \cap B \neq \emptyset$, *and* $u_k(t) = v_k(t)$ *for every point* t *in* $A \cap B$ *and* $k = 0, ..., n$. *Assume further that for every* $c \in A \cap B$, *every point in* A *to the right of* c *lies in* B, *and every point of* B *to the left of* c *lies in* A. *Then there are weakly nondegenerate weak Markov systems* $\tilde{U}_n$ *and* $\tilde{V}_n$ *defined respectively on intervals* I_1 *and* I_2 *whose union is open, and a strictly increasing function* $h: A \cup B \to I_1 \cup I_2$, *such that all the functions in* $\tilde{U}_n$ *are C-absolutely continuous on* I_1, *all the functions in* $\tilde{V}_n$ *are C-absolutely continuous on* I_2, *and we have*:

(a) $h(t)$ *embeds* U_n *into* $\tilde{U}_n$,

(b) $h(t)$ *embeds* V_n *into* $\tilde{V}_n$,

(c) *for* $k = 0, ..., n$, $\tilde{u}_k(t) = \tilde{v}_k(t)$ *on* $I_1 \cap I_2$.

Proof. Since the method of proof is similar to that of [8, Theorem 3], some details will be omitted.

For $k = 0, ..., n$, let $z_k(t) := u_k(t)$ if $t \in A$, and $z_k(t) := v_k(t)$ if $t \in B$. Clearly, $z_1(t)$ is increasing. Let $C := A \cup B$, $l_1 := \inf(C)$, $l_2 := \sup(C)$, and let $S = \{s_i\}$ denote the set of points of accumulation of C at which $z_1(t)$ has jump discontinuities. If $s_i \in S$, let $d_i = 2^{-i}$; if in addition $s_i \in C$, let

$a_i = 2^{-(i+1)}$ if $z_1(s_i^+) - z_1(s_i) > 0$, and 0 otherwise. We are now able to define a strictly increasing function q on C in the following manner:

$$q(t) := t + \sum_{s_j < t} d_j \qquad \text{for all } t \in C \backslash S,$$

$$q(t_i) = t + a_i + \sum_{s_j < t_i} d_j \qquad \text{at any point } t_i \text{ in } C \cap S.$$

Setting $z_k^{(0)}(t) := z_k[q^{-1}(t)]$, we infer from [9, Lemma 1] that $Z_n^{(0)}$ is weakly nondegenerate when restricted either to $A_0 := q[A]$ or to $B_0 := q[B]$. Moreover, the function $z_1^{(0)}$ is either continuous or has a removable discontinuity at every point of accumulation of $C_0 := q[C]$.

As in the proof of [9, Theorem 3], we see that there is no loss of generality in assuming that $\inf(A) \notin A$ and $\sup(B) \notin B$. Let $l_1^{(0)} := \inf(C_0)$, $l_2^{(0)} := \sup(C_0)$, and let $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$ denote the closures of A_0, B_0, C_0 in the relative topology of $I := (l_1^{(0)}, l_2^{(0)})$. We shall now extend the domain of definition of $Z_n^{(0)}$ to $\bar{C}_0$. We divide the argument into four subcases. Note that, for every point c in $A_0 \cap B_0$, every point in B_0 to the left of c is in A_0, and every point in A_0 to the right of c is in B_0.

Case I. The point t is in C_0. In this case, we define $z_k^{(1)}(t) := z_k^{(0)}(t)$ for $k = 0, \ldots, n$.

Case II. The point t is in $\bar{A}_0 \backslash A_0$ but not in $\bar{B}_0$. Then

$$z_k^{(1)}(t) := \lim_{x \to t^-} z_k^{(0)}(x)$$

if t is a point of accumulation of $(l_1, t) \cap A_0$; otherwise

$$z_k^{(1)}(t) := \lim_{x \to t^+} z_k^{(0)}(x).$$

That this can be done follows from [9, Lemma 3].

Case III. The point t is in $\bar{B}_0 \backslash B_0$ but not in $\bar{A}_0$. Then

$$z_k^{(1)}(t) := \lim_{x \to t^+} z_k^{(0)}(x)$$

if t is a point of accumulation of $(t, l_2) \cap B_0$; otherwise

$$z_k^{(1)}(t) := \lim_{x \to t^-} z_k^{(0)}(x).$$

Case IV. The point t is in $\bar{A}_0 \cap \bar{B}_0$ but not in $A_0 \cap B_0$. There are four subcases.

(a) If t is a point of accumulation of $(l_1, t) \cap A_0$ and also of $(l_1, t) \cap B_0$, we define

$$z_k^{(1)}(t) := \lim_{x \to t^-} z_k^{(0)}(x).$$

(b) If t is a point of accumulation of $(t, l_2) \cap A_0$ and of $(t, l_2) \cap B_0$ and has not been considered in IV(a), we set

$$z_k^{(1)}(t) := \lim_{x \to t^+} z_k^{(0)}(x).$$

(c) If t is a point of accumulation of $(l_1, t) \cap A_0$ and of $(t, l_2) \cap B_0$ then, since $A_0 \cap B_0 \neq \emptyset$ and $t \notin A_0 \cap B_0$, there must be a point in $A_0 \cap B_0$ lying either to the left of t or to the right of t. In the first case, we deduce from the hypotheses that t is also a point of accumulation of $(l_1, t) \cap B_0$, and we return to IV(a). Otherwise, t must be a point of accumulation of $(t, l_2) \cap A_0$, and we return to IV(b).

(d) Finally, if t is a point of accumulation of $(l_1, t) \cap B_0$ and of $(t, l_2) \cap A_0$, we readily deduce that it must also be a point of accumulation of $(l_1, t) \cap A_0$ or of $(t, l_2) \cap B_0$, and we again return to either IV(a) or IV(b).

The previous construction guarantees that $Z_n^{(1)}$ is a normalized weak Markov system when restricted to either of the sets $\bar{A}_0$ or $\bar{B}_0$. Proceeding as in the proof of [9, Theorem 3], it is also readily seen that $Z_n^{(1)}$ is weakly nondegenerate on each of the sets $\bar{A}_0$ and $\bar{B}_0$. Moreover, for every point c in $\bar{A}_0 \cap \bar{B}_0$, every point of $\bar{B}_0$ to the left of c lies in $\bar{A}_0$, and every point of $\bar{A}_0$ to the right of c lies in $\bar{B}_0$. Indeed, assume, e.g., that $c \in \bar{A}_0 \cap \bar{B}_0$, $t < c$, and $t \in \bar{B}_0$, let $\{t_n\}$ be a sequence of points in B_0 converging to t, and let $c_1 \in A_0 \cap B_0$. If $c_1 < c$, then every point in $A_0 \cap (c_1, c)$ is in B_0. Thus, $c \in \overline{A_0 \cap B_0}$, and we conclude that there are points in $\overline{A_0 \cap B_0}$ to the right of t; thus for n large enough t_n is to the left of a point in $A_0 \cap B_0$, and therefore $t_n \in A_0$. If $c_1 > c$, then for n large enough, $t_n < c_1$ and therefore $t \in A_0$ in this case as well. We thus conclude that $t \in \bar{A}_0$.

Clearly, the complementary set of $\bar{C}_0$ in $(l_1^{(0)}, l_2^{(0)})$, if not empty, is a disjoint union of open intervals $G_j = (c_j, d_j)$. The hypotheses allow for the following possibilities, writing $m := \inf(\bar{A}_0 \cap \bar{B}_0)$ and $M := \sup(\bar{A}_0 \cap \bar{B}_0)$.

(a) Both c_j and d_j are in $\bar{B}_0 \backslash \bar{A}_0$, or in $\bar{A}_0 \backslash \bar{B}_0$,

(b) both c_j and d_j are in $\bar{A}_0 \cap \bar{B}_0$,

(c) $c_j \in \bar{A}_0$ and $d_j = m$ (whence $d_j \in \bar{A}_0 \cap \bar{B}_0$ and $c_j \in \bar{A}_0 \backslash \bar{B}_0$),

(d) $d_j \in \bar{B}_0$ and $c_j = M$ (whence $c_j \in \bar{A}_0 \cap \bar{B}_0$ and $d_j \in \bar{B}_0 \backslash \bar{A}_0$).

By linear interpolation, as in the proof of [9, Theorem 3], it is therefore easy to see that there is a system W_n and two overlapping intervals I_1 open to the left and I_2 open to the right such that:

(a) W_n is a weakly nondegenerate weak Markov system when restricted to either I_1 or I_2,

(b) the identity function embeds the restriction of $Z_n^{(1)}$ to $\bar{A}_0$ into the restriction of W_n to I_1 and the restriction of $Z_n^{(1)}$ to $\bar{B}_0$ into the restriction of W_n into I_2,

(c) W_n is continuous on $I := I_1 \cup I_2$ (which is open).

The rest of the proof is almost identical to that of the corresponding part of [9, Theorem 3] and can be omitted. This therefore completes the proof of Lemma 1.

The following assertion was made in the proof of [9, Theorem 1], with a brief outline of its proof. It is appropriate to give a more detailed proof, since in this context also the assertion is central to the argument.

LEMMA 2. *Let U_n be a weakly nondegenerate normalized weak Markov system of C-absolutely continuous functions defined on an interval (a, b), and let D be the subset of (a, b) on which all the functions are differentiable. Then $U'_n := \{u'_1, ..., u'_n\}$ is a weakly nondegenerate weak Markov system on D.*

Remark 4. We will take as a starting point for our proof of the Lemma that the derived system U'_n is a weak Markov system on its domain, D. The proof of this statement is almost identical to that of [5, Lemma 3.1] and need not be repeated here (see also the proof of [11, Theorem 11.3(b)]).

Proof of Lemma 2. We proceed with the proof that the system U'_n is weakly nondegenerate.

Let I be a subinterval of (a, b) of the form $(a, c]$ or $[c, b)$, and let $\{k(r): r = 0, ..., m\}$ be a subsequence of $\{0, ..., n\}$ with $k(0) = 0$. Since

$$u_k(t) = u_k(c) + \int_c^t u'_k(s)\, ds,$$

it is readily seen that if the system $\{u'_{k(r)}: r = 1, ..., m\}$ is linearly dependent on $I \cap D$ then $\{u_{k(r)}: r = 0, ..., m\}$ is linearly dependent on I. Indeed, there are numbers a_i, not all zero, such that $a_1 u'_{k(1)} + \cdots + a_m u'_{k(m)} = 0$ on $I \cap D$. Thus on I we have $a_1 u_{k(1)}(t) + \cdots + a_m u_{k(m)}(t) = a_1 u_{k(1)}(c) = \cdots + a_m u_{k(m)}(c)$. Since the right-hand member of this equation is a constant, the proof of the asserted linear dependence follows.

From this argument, we conclude that U'_n satisfies Condition I. For, if c is any point in (a, b), linear dependence of U'_n on both $(a, c] \cap D$ and on

$[c, b) \cap D$ would imply linear dependence of U_n on both of the sets $(a, c]$ and $[c, b)$, violating Condition I for U_n.

Assume now that, for some point c in (a, b), U_n' is linearly independent on $[c, b) \cap D$. Since, as is easy to show, U_n is linearly independent on $[c, b)$, we infer from Condition E that there is a set V_n, obtained from U_n by a triangular transformation, such that for every subsequence $\{k(r): r = 0, ..., m\}$ of the sequence $\{0, ..., n\}$ for which $k(0) = 0$, $\{v_{k(r)}: r = 0, ..., m\}$ is a weak Markov system on $[c, b)$. As noted in Remark 4, we may conclude that $\{v'_{k(r)}: r = 1, ..., m\}$ is a weak Markov system on $[c, b) \cap D$. We have therefore proved that U_n' satisfies Condition E(i). The proof of Condition E(ii) is similar and will be omitted.

Proof of Theorem 1. Let U_n and V_n be weak Markov systems defined respectively on sets A and B, such that the hypotheses of the theorem are satisfied. Our proof will proceed by induction on n. We note that the theorem is certainly true if $n = 0$ or if $n = 1$. We devote our attentions therefore to the general case.

Applying Lemma 1, we conclude that there are weakly nondegenerate normalized weak Markov systems $\tilde{U}_n$ and $\tilde{V}_n$ defined respectively on intervals I_1 and I_2 with $I := I_1 \cup I_2$ open, and a strictly increasing function $h: A \cup B \to I$ such that all the elements of $\tilde{U}_n$ are C-absolutely continuous on I_1, and all the elements of $\tilde{V}_n$ are C-absolutely continuous on I_2.

The set of functions Z_n defined in the hypotheses of our theorem by $z_k(t) = u_k(t)$ for t in A and $k = 0, ..., n$ and by $z_k(t) = v_k(t)$ for t in B and $k = 0, ..., n$ can now be seen to map to a system of C-absolutely continuous functions $\tilde{Z}_n$ defined on I and satisfying $z_k(t) = \tilde{z}_k(h(t))$ for all t in $A \cup B$. Clearly, the system Z_n will be a weakly nondegenerate weak Markov system if the system $\tilde{Z}_n$ is, and we proceed to demonstrate this fact. The argument parallels that of [9, Theorem 1].

As observed above, the functions in the set $\tilde{Z}_n$ are C-absolutely continuous. Thus, there exists a set D whose measure is equal to that of I on which all of the functions in $\tilde{Z}_n$ are differentiable, and, defining $D_j = D \cap I_j$ for $j = 1, 2$, we consider the systems of derivatives $\tilde{U}_n'$ defined on D_1, $\tilde{V}_n'$ defined on D_2, and $\tilde{Z}_n'$ defined on D. Applying Lemma 2, we note that $\tilde{U}_n'$ is a weakly nondegenerate weak Markov system on D_1. In like manner, we see that V_n' is a weakly nondegenerate weak Markov system on D_2.

Now, in view of the fact that $\tilde{u}_1$ is a nonconstant and increasing function, there is a set $E_1 \subseteq D_1$ on which $\tilde{u}_1' > 0$. Since the function $\tilde{u}_1$ is C-absolutely continuous, the set E_1 is necessarily of positive measure. In like manner, there is a set $E_2 \subseteq D_2$ of positive measure on which $\tilde{v}_1' > 0$. Moreover, in view of the fact that U_n (and thus also V_n) is linearly independent on $A \cap B$ by hypothesis, it follows that $\tilde{U}_n$ and $\tilde{V}_n$ are linearly independent on $I_1 \cap I_2$, which must therefore be a nondegenerate interval containing

$E_1 \cap E_2$, which must again be a set of positive measure and in particular must be nonempty.

We are now ready to begin the induction step. We define $P := \{p_1, ..., p_n\}$ on E_1 by $p_k := \tilde{u}'_k/\tilde{u}'_1$ for $k = 1, ..., n$; $Q := \{q_1, ..., q_n\}$ on E_2 by $q_k := \tilde{v}'_k/\tilde{v}'_1$ for $k = 1, ..., n$; and $R := \{r_1, ..., r_n\}$ on $E := E_1 \cup E_2$ by $r_k := \tilde{z}'_k/\tilde{z}'_1$ for $k = 1, ..., n$. The systems P and Q are seen to be weakly nondegenerate normalized weak Markov systems which are defined on the overlapping sets E_1 and E_2. At any point d in the intersection of E_1 and E_2, it is true that any point of E_1 lying to the right of d is an element of E_2, and any point of E_2 lying to the left of d is in E_1. Both P and Q are linearly independent on the intersection of E_1 and E_2. Thus, the system R is a weakly nondegenerate normalized weak Markov system on E, by the induction hypothesis.

We now complete the proof by showing that $\tilde{Z}_n$ is a weakly nondegenerate normalized weak Markov system on I. We begin by noting that $\tilde{Z}'_n$ is a weakly nondegenerate weak Markov system on E because R is a weakly nondegenerate normalized weak Markov system. We note that, if t is in $D \setminus E$, then either $t \in D_1$ or $t \in D_2$. If the first, then $\tilde{u}'_1(t) = 0$, and from Theorem 3 we see that $\tilde{z}'_k(t) = \tilde{u}'_k(t) = 0$ for every t in $D_1 \setminus E$, for $k = 1, ..., n$. If on the other hand the point t lies in D_2, we consider the system $\tilde{V}'_n$. In either case, we conclude that $\tilde{z}'_k(t) = 0$ for all k, $k = 1, ..., n$. Therefore $\tilde{Z}'_n$ is a weak Markov system on all of D, a fact which implies that $\tilde{Z}_n$ is a weakly nondegenerate normalized weak Markov system on the interval I, for we may then write for t in I and for $k = 1, ..., n$,

$$\tilde{z}_k(t) = \tilde{z}_k(c) + \int_c^t \tilde{z}'_k(t_1)\, dt_1,$$

where c is an arbitrary point in $I_1 \cap I_2$. This completes the proof of Theorem 1.

Proof of Theorem 2. As we have previously remarked, [10, Theorem 1] implies that U_n and V_n satisfy Condition E and are therefore weakly nondegenerate. Thus, from Theorem 1 we know that Z_n is a weak Markov system on $A \cup B$. Since $z_0(t)$ is strictly positive there, it follows from Theorem 4 that Z_k is a Tchebycheff system for $k = 0, ..., n$. Hence, Z_n is in fact a Markov system, and we have demonstrated Theorem 2.

An Example

We include one simple example to show what can happen if the hypotheses of Theorem 1 or Theorem 2 are relaxed. Let the system U_2 be defined by $u_0 = 1$, $u_1(t) = t$, $u_2(t) = t^2$, on the set $A = [0, 1]$. The system V_2

will be defined on the set $B=[1,2]$ by $v_0=1$, $v_1(t)=t$, $v_2(t)=(t-1)^2+1$. The set Z_2 defined by the splicing of the functions in U_2 and V_2 on the interval $[0,2]$ fails to be a weak Markov system. Note that the intersection of the sets A and B consists of just one point.

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